

Algebraic and analytic properties of quasimetric spaces with dilations*

Svetlana Selivanova, Sergei Vodopyanov

Abstract

We provide an axiomatic approach to the theory of local tangent cones of regular sub-Riemannian manifolds and the differentiability of mappings between such spaces. This axiomatic approach relies on a notion of a dilation structure which is introduced in the general framework of quasimetric spaces. Considering quasimetrics allows us to cover a general case including, in particular, minimal smoothness assumptions on the vector fields defining the sub-Riemannian structure. It is important to note that the theory existing for metric spaces can not be directly extended to quasimetric spaces.

Key words: Dilations, local group, contractible group, Mal'tsev's theorem, tangent cone, Carnot-Carathéodory space, differentiability

MSC: Primary 22E05, 53C17; Secondary 20F17, 22D05, 54E50.

1 Introduction

We study algebraic and analytic properties of quasimetric spaces endowed with dilations (roughly speaking, dilations are continuous one-parameter families of contractive homeomorphisms given in a neighborhood of each point).

Our work is motivated by investigation of metric properties of Carnot-Carathéodory spaces, also referred to as sub-Riemannian manifolds which model nonholonomic processes and naturally arise in many applications (see e. g. [1, 2, 5, 11, 12, 25, 27, 18, 29, 32, 36, 39, 45, 49] and references therein).

Let us first recall the “classical” definition of a sub-Riemannian manifold. Given a smooth connected manifold \mathbb{M} of dimension N and smooth “horizontal” vector fields $X_1, \dots, X_m \in C^\infty$ on \mathbb{M} (where $m \leq N$), it is assumed that these vector fields span, together with their commutators, the tangent space to \mathbb{M} at each point (Hörmander's condition [27]). By Rashevskii-Chow's Theorem, any two points of \mathbb{M} can be connected by a horizontal curve and, therefore, there exists an intrinsic sub-Riemannian metric d_c on \mathbb{M} defined as the infimum over lengths of all horizontal curves.

Recently discovered applications have lead to considering a more general situation [28, 29, 46, 54, 55, 56] when

- 1) a maximal possible reduction of smoothness of the vector fields is made (see also [4, 22, 35]);

*This research was partially supported by Federal Target Grant "Scientific and educational personnel of innovation Russia" for 2009-2013 (government contract No. P2224) and by the program "Leading Scientific Schools" (project N. NSh-5682.2008.1).

2) instead the Hörmander's condition, a weaker one of a “weighted” filtration of TM (see Definition 10) is assumed (see also [17, 18, 22, 39, 49]).

Under these general assumptions, the intrinsic metric d_c might not exist, but a certain quasimetric (a distance function meeting a generalized triangle inequality, see Definition 1) can be introduced (see [39] where various quasimetrics induced by families of vector fields on \mathbb{R}^N were studied).

On the other hand, recent development of analysis on general metric spaces has lead to the question of describing the most general approach to the metric geometry of sub-Riemannian manifolds. Among possible approaches is considering metric spaces with dilations [2, 6, 9, 18].

Motivated by these considerations, we extend the notion of a dilation structure to quasimetric spaces and investigate local properties of the obtained object.

In 1981 M. Gromov has defined [23, 24] the tangent cone to a metric space (X, d) at a point $x \in X$ as the limit of pointed scaled metric spaces $(X, x, \lambda \cdot d)$ (when $\lambda \rightarrow \infty$) w. r. t. Gromov-Hausdorff distance. This notion generalizes the concept of the tangent space to a manifold and is useful in the general theory of metric spaces (see e. g. [3, 11, 13, 15, 43]), in particular, Carnot-Carathéodory spaces [32, 34].

A straightforward generalization of Gromov's theory would make no sense for quasimetric spaces, see Remark 6. In [46, 47] a convergence theory for quasimetric spaces with the following properties was developed:

- 1) it includes the Gromov-Hausdorff convergence for metric spaces as a particular case;
- 2) the limit is unique up to isometry for boundedly compact quasimetric spaces;
- 3) it allows to introduce the notion of the tangent cone in the same way as for metric spaces.

In [47] the existence of the tangent cone (w. r. t. the introduced convergence) to a quasimetric space with dilations is proved (see Definition 2, Axioms (A0) — (A3), and Theorem 2). This statement contains as a particular case a similar result by M. Buliga for metric spaces, see for instance [6], where an axiomatic approach to metric spaces with dilations is introduced. A similar approach was informally sketched by A. Bellaïche [2].

The main results of the present paper are Theorems 4 and 7. Theorem 4 (cf. [7]) asserts that an additional axiom (A4) (saying that the limit of a certain combination of dilations exists) allows to describe the algebraic structure of the tangent cone: it is a simply connected Lie group, the Lie algebra of which is graded and nilpotent.

In particular, this result allows to define the differential of a mapping acting between two quasimetric spaces with dilations in the same way as it is done in [50] for Carnot-Carathéodory spaces. A brief comparison of this approach with Margulis-Mostow's concept of differentiability [32] is given below in Remark 14.

Thus, Theorem 4 allows to establish algebraic and analytic properties of the considered space from metric and topological assumptions only. In the present paper we do not attempt to prove that axioms of a dilation structure recover sub-Riemannian geometry when the underlying space is a manifold (or which axioms should be added to prove this). But we prove that

- 1) regular sub-Riemannian manifolds are examples of quasimetric spaces with dilations (Theorem 7);
- 2) the tangent cones to quasimetric spaces with dilations are the same algebraic objects as for regular sub-Riemannian manifolds (Theorem 4),
which can be viewed as a first step in this direction.

In our opinion, the proof of Theorem 4 is interesting in its own right. The main step is to apply a theorem on local and global topological groups due to A. I. Mal'tsev [31], which helps to overcome difficulties concerned with investigation of a local version of the Hilbert's Fifth Problem [58, 19, 37], see Remark 2. As an auxiliary assertion we prove a generalized triangle inequality for local groups endowed with (quasi)metrics and dilations (see Proposition 8, Assertion 3)), which is of independent interest and gives an alternative proof of a similar fact for (global) homogeneous groups [18].

In Section 4, we describe regular Carnot-Carathéodory spaces as the main example of quasimetric spaces with dilations. In this case Axiom (A3) is just a local approximation theorem, and (A4) is a consequence of estimates on divergence of integral lines of the initial vector fields and the nilpotentized ones.

In this paper we extend the approach to the subject given in our short communication [57].

We are grateful to Isaac Goldbring for a discussion on some algebraic aspects of the subject under consideration (see Remark 9) and for the references [40, 20]. We thank also the anonymous referee for the careful reading of our paper, interesting questions and references, as well as useful hints concerning the presentation and exposition of our results.

2 Basic notions and preliminary results

Definition 1. A *quasimetric space* $(\mathbb{X}, d_{\mathbb{X}})$ is a topological space \mathbb{X} with a quasimetric $d_{\mathbb{X}}$. A *quasimetric* is a mapping $d_{\mathbb{X}} : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}^+$ with the following properties:

- (1) $d_{\mathbb{X}}(u, v) \geq 0$; $d_{\mathbb{X}}(u, v) = 0$ if and only if $u = v$ (non-degeneracy);
- (2) $d_{\mathbb{X}}(u, v) \leq c_{\mathbb{X}} d_{\mathbb{X}}(v, u)$ where $1 \leq c_{\mathbb{X}} < \infty$ is a constant independent of $u, v \in \mathbb{X}$ (generalized symmetry property);
- (3) $d_{\mathbb{X}}(u, v) \leq Q_{\mathbb{X}}(d_{\mathbb{X}}(u, w) + d_{\mathbb{X}}(w, v))$ where $1 \leq Q_{\mathbb{X}} < \infty$ is a constant independent of $u, v, w \in \mathbb{X}$ (generalized triangle inequality);
- (4) the function $d_{\mathbb{X}}(u, v)$ is upper semi-continuous on the first argument.

If $c_{\mathbb{X}} = 1$, $Q_{\mathbb{X}} = 1$, then $(\mathbb{X}, d_{\mathbb{X}})$ is a metric space.

Remark 1. Note that some authors introduce the notion of a quasimetric space without assuming neither this space be topological nor the quasimetric be continuous in any sense. Within such framework, the quasimetric balls need not be open (see e. g. [41, 14, 26]). However, due to a theorem by R. A. Macías and C. Segovia [30], any quasimetric d is equivalent to some other quasimetric \tilde{d} , the balls associated to which are open (such a quasimetric looks like $\rho(x, y)^{\frac{1}{\beta}}$, where $0 < \beta \leq 1$ and $\rho(x, y)$ is a metric) and, hence, define a topology.

In the present paper we study tangent cone questions. It is important to note, that having the tangent cone to a (quasi)metric space, one can say nothing about the existence of the tangent cone to the space with an equivalent (quasi)metric, thus we would like the balls defined by the initial quasimetric be open. For this reason we add the upper-continuity condition (4) to the Definition 1 of a quasimetric space (as it is done e. g. in [49] for the case of \mathbb{R}^n). This condition guarantees that the balls $B^{d_{\mathbb{X}}}(x, r)$ are open sets, and that convergence w. r. t. the initial topology of \mathbb{X} implies convergence w. r. t. the topology defined by $d_{\mathbb{X}}$.

Actually, we can assume the initial topology on \mathbb{X} coincide with the topology induced by the equivalent quasimetric \tilde{d} . Then the topologies induced by d and convergence w. r. t. initial topology on \mathbb{X} are equivalent. Further we always assume, w. l. o. g., this to hold.

We denote by $B^{d_{\mathbb{X}}}(x, r) = \{y \in \mathbb{X} \mid d_{\mathbb{X}}(y, x) < r\}$ a ball centered at x of radius r , w. r. t. the (quasi)metric $d_{\mathbb{X}}$. The symbol \bar{A} stands for the closure of the set A . A (quasi)metric space \mathbb{X} is said to be *boundedly compact* if all closed bounded subsets of \mathbb{X} are compact.

Definition 2. Let (\mathbb{X}, d) be a complete boundedly compact quasimetric space and the quasimetric d be continuous on both arguments. The quasimetric space \mathbb{X} is endowed with a *dilation structure*, denoted as (\mathbb{X}, d, δ) , if the following axioms (A0) — (A3) hold.

(A0) For all $x \in \mathbb{X}$ and for $\varepsilon \in (0, 1]$, in some neighborhood $U(x)$ of x there are homeomorphisms called *dilations* $\delta_{\varepsilon}^x : U(x) \rightarrow V_{\varepsilon}(x)$ and $\delta_{\varepsilon^{-1}}^x : W_{\varepsilon^{-1}}(x) \rightarrow U(x)$, where $V_{\varepsilon}(x) \subseteq W_{\varepsilon^{-1}}(x) \subseteq U(x)$. The family $\{\delta_{\varepsilon}^x\}_{\varepsilon \in (0, 1]}$ is continuous on ε (w. r. t. the initial topology on \mathbb{X} , see Remark 1, and the ordinary topology on $(0, 1]$). It is assumed that there exists an $R > 0$ such that $\bar{B}^d(x, R) \subseteq U(x)$ for all $x \in \mathbb{X}$, and for all $\varepsilon < 1$ and $\tilde{r} > 0$ with the property $\bar{B}^d(x, \tilde{r}) \subseteq U(x)$ we have the inclusion $B^d(x, \tilde{r}\varepsilon) \subseteq \delta_{\varepsilon}^x B^d(x, \tilde{r}) \subset B^d(x, \tilde{r})$.

(A1) For all $x \in \mathbb{X}$, $y \in U(x)$, we have $\delta_{\varepsilon}^x x = x$, $\delta_1^x = \text{id}$, $\lim_{\varepsilon \rightarrow 0} \delta_{\varepsilon}^x y = x$.

(A2) For all $x \in \mathbb{X}$ and $u \in U(x)$, we have $\delta_{\varepsilon}^x \delta_{\mu}^x u = \delta_{\varepsilon\mu}^x u$ provided that both parts of this equality are defined.

(A3) For any $x \in \mathbb{X}$, uniformly on $u, v \in \bar{B}^d(x, R)$ there exists the limit

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} d(\delta_{\varepsilon}^x u, \delta_{\varepsilon}^x v) = d^x(u, v). \quad (2.1)$$

If the function $d^x : U(x) \times U(x) \rightarrow \mathbb{R}^+$ is such that $d^x(u, v) = 0$ implies $u = v$, then the dilation structure is called *nondegenerate*.

If the convergence in (A3) is uniform on x in some compact set, then the dilation structure is said to be *uniform*.

If the following axiom (A4) holds, then we say that \mathbb{X} is endowed with a *strong dilation structure*.

(A4) The limit of the value $\Lambda_{\varepsilon}^x(u, v) = \delta_{\varepsilon^{-1}}^x \delta_{\varepsilon}^x v$ exists:

$$\lim_{\varepsilon \rightarrow 0} \Lambda_{\varepsilon}^x(u, v) = \Lambda^x(u, v) \in B^d(x, R), \quad (2.2)$$

This limit is uniform on x in some compact set and $u, v \in B^d(x, r)$ for some $0 < r \leq R$. See Proposition 4 regarding possible choices of r .

Remark 2. These axioms of dilations are a slight modification and simplification of those proposed in [6] for metric spaces. Essential for proving Theorem 4 is that, in (A0), we require the continuity of dilations on the parameter ε which was missed in [6]. Note also that axioms (A1), (A2), (A4) do not depend on the quasimetric. The condition $\lim_{\varepsilon \rightarrow 0} \delta_{\varepsilon}^x y = x$ informally states that the topological space \mathbb{X} is locally contractible.

Example 1. In the case when \mathbb{X} is a Riemannian manifold, dilations can be introduced as homotheties induced by the Euclidean ones. See [6]–[10] for more examples.

Remark 3. For a general (quasi)metric space (\mathbb{X}, d) , the closure of a ball need not coincide with the corresponding closed ball, only the inclusion $\bar{B}^d(x, r) \subseteq \{y : d(y, x) \leq r\}$ holds. But, in the case of a (quasi)metric space endowed with a dilation structure, also the converse inclusion is true. Indeed, let $z \in \{y : d(y, x) \leq r\}$ be such that $d(z, x) = r$; let $z_n = \delta_{1-\varepsilon_n}^x z \in B^d(x, r)$, where $\varepsilon_n \rightarrow 0$. Then $d(z_n, z) \rightarrow 0$, according to (A0), (A1) and Remark 1, hence $z \in \bar{B}^d(x, r)$.

Remark 4. By virtue of (A3) and continuity of $d(u, v)$, the function $d^x(u, v)$ is continuous on both arguments. Further, the functions d^x and d define the same topology on $U(x)$ (the equivalence of convergences induced by d^x and d can be verified straightforwardly, using uniformity on u, v in (A3)) and, hence, $(U(x), d^x)$ is boundedly compact.

Remark 4 and Axiom (A3) imply

Proposition 1. *If (\mathbb{X}, d, δ) is a nondegenerate dilation structure, then d^x is a quasimetric on $B^d(x, R)$ with the same constants $c_{\mathbb{X}}, Q_{\mathbb{X}}$ (see (2), (3) of Definition 1) as for the initial quasimetric d .*

In the same way as for metric spaces [6], Axioms (A2), (A3) imply

Proposition 2. *The function d^x from Axiom (A3) meets the cone property*

$$d^x(u, v) = \frac{1}{\mu} d^x(\delta_{\mu}^x u, \delta_{\mu}^x v)$$

for all $u, v \in B^d(x, R)$ and μ such that $\delta_{\mu}^x u, \delta_{\mu}^x v \in B^d(x, R)$ (in particular, for all $\mu \leq 1$).

Proposition 3. *If (\mathbb{X}, d, δ) is a strong dilation structure then the limits of the expressions $\Sigma_{\varepsilon}^x(u, v) = \delta_{\varepsilon^{-1}}^x \delta_{\varepsilon}^{\delta_{\varepsilon}^x u} v$, $\text{inv}_{\varepsilon}^x(u) = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^x u} x$ exist:*

$$\lim_{\varepsilon \rightarrow 0} \Sigma_{\varepsilon}^x(u, v) = \Sigma^x(u, v) \in B^d(x, R), \quad \lim_{\varepsilon \rightarrow 0} \text{inv}_{\varepsilon}^x(u) = \text{inv}^x(u) \in B^d(x, R). \quad (2.3)$$

These limits are uniform on x in some compact set and $u, v \in B^d(x, \hat{r})$.

Conversely, if the limits 2.3 exist and are uniform, then Axiom (A4) holds.

Proof. The assertion about the second limit follows from the fact that $\text{inv}_{\varepsilon}^x(u) = \Lambda_{\varepsilon}^x(u, x)$. Easy calculations show that $\Sigma_{\varepsilon}^x(u, v) = \Lambda_{\varepsilon}^{\delta_{\varepsilon}^x u}(\text{inv}_{\varepsilon}^x u, v)$ from where, taking in account the uniformity of convergence in (A4), the existence and uniformity of the first limit follows.

Moreover, it is easy to see that $\Sigma_{\varepsilon}^{\delta_{\varepsilon}^x u}(\text{inv}_{\varepsilon}^x u, v) = \Lambda_{\varepsilon}^x(u, v)$, hence

$$\Lambda^x(u, v) = \Sigma^x(\text{inv}^x u, v). \quad (2.4)$$

Therefore, from the existence and uniformity of the limits 2.3, Axiom (A4) follows. \square

Further we assume, w. l. o. g., that $\hat{r} = r$ (otherwise, take the intersection of the corresponding balls), i. e. functions Λ^x and Σ^x are defined on the same subset of $U(x) \times U(x)$. The following proposition can be viewed as an example of existence of one of the combinations from Proposition 3 (cf. the arguments of Bellaïche [2], the last section).

Proposition 4. *Let (\mathbb{X}, d, δ) be a uniform dilation structure. Then there are $r, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, $u, v \in B^d(x, r)$ the combination $\Sigma_{\varepsilon}^x(u, v) = \delta_{\varepsilon^{-1}}^x \delta_{\varepsilon}^{\delta_{\varepsilon}^x u} v \in U(x)$ from Proposition 3 is defined.*

Proof. Let $x' = \delta_\varepsilon^x u$, $x'' = \delta_\varepsilon^{x'} v$. To show the existence of the combination $\Sigma_\varepsilon^x(u, v) \in U(x)$ it suffices to verify that $x'' \in W_{\varepsilon^{-1}}(x)$. Let us prove that, for suitable u, v, ε , it is true that $x'' \in B^d(x, R\varepsilon) \subseteq W_{\varepsilon^{-1}}(x)$. It follows from Proposition 2 that $d^x(x, x') = d^x(x, \delta_\varepsilon^x u) = \varepsilon d^x(x, u)$, $d^{x'}(x', x'') = \varepsilon d^{x'}(x', v)$. Due to (A3), for any $\delta > 0$ there is an $\varepsilon > 0$ such that: if $d^x(p, q) = O(\varepsilon)$, then $d^x(p, q)(1 - \delta) \leq d(p, q) \leq d^x(p, q)(1 + \delta)$. Let $p = x$, $q = x'$ and consider arbitrary $r, R^x > 0$ such that $B^d(x, r) \subseteq B^{d^x}(x, R^x) \subseteq B^d(x, R)$ (such reals exist according to Remark 4). For any $\delta > 0$ there is an $\varepsilon'_0 > 0$ such that for $u \in B^d(x, r)$, $\varepsilon \in (0, \varepsilon'_0]$ we have $d(x, x') \leq \varepsilon R^x(1 + \delta)$. Analogously, there is an $\varepsilon''_0 > 0$ such that for $v \in B^d(x, r)$, $\varepsilon \in (0, \varepsilon''_0]$ we have $d(x', x'') \leq \varepsilon R^{x'}(1 + \delta)$. Due to uniformity of the limit in (A3) we can assume, w. l. o. g., that $R^x = R^{x'} = \xi$. Let $\varepsilon_0 = \min\{\varepsilon'_0, \varepsilon''_0\}$. The generalized triangle inequality implies $d(x, x'') \leq Q_{\mathbb{X}}(d(x, x') + d(x', x'')) \leq 2Q_{\mathbb{X}}\varepsilon\xi(1 + \delta)$. To satisfy the desired inequality $d(x, x'') \leq R\varepsilon$ it suffices to take an arbitrary $\xi < \frac{R}{2Q_{\mathbb{X}}}$ such that $B^{d^x}(x, \xi) \subseteq B^d(x, R)$. Then an arbitrary number r satisfying $B^d(x, r) \subseteq B^{d^x}(x, \xi)$ will be as desired. \square

A *pointed (quasi)metric space* is a pair (\mathbb{X}, p) consisting of a (quasi)metric space \mathbb{X} and a point $p \in \mathbb{X}$. Whenever we want to emphasize what kind of (quasi)metric is on \mathbb{X} , we shall write the pointed space as a triple $(\mathbb{X}, p, d_{\mathbb{X}})$.

Definition 3 ([46, 47]). A sequence $(\mathbb{X}_n, p_n, d_{\mathbb{X}_n})$ of pointed quasimetric spaces *converges* to the pointed space $(\mathbb{X}, p, d_{\mathbb{X}})$, if there exists a sequence of reals $\delta_n \rightarrow 0$ such that for each $r > 0$ there exist mappings $f_{n,r} : B^{d_{\mathbb{X}_n}}(p_n, r + \delta_n) \rightarrow \mathbb{X}$, $g_{n,r} : B^{d_{\mathbb{X}}}(p, r + 2\delta_n) \rightarrow \mathbb{X}_n$ such that

- 1) $f_{n,r}(p_n) = p$, $g_{n,r}(p) = p_n$;
- 2) $\text{dis}(f_{n,r}) < \delta_n$, $\text{dis}(g_{n,r}) < \delta_n$;
- 3) $\sup_{x \in B^{d_{\mathbb{X}_n}}(p_n, r + \delta_n)} d_{\mathbb{X}_n}(x, g_{n,r}(f_{n,r}(x))) < \delta_n$.

Here $\text{dis}(f) = \sup_{u, v \in \mathbb{X}} |d_{\mathbb{Y}}(f(u), f(v)) - d_{\mathbb{X}}(u, v)|$ is the *distortion* of a mapping $f : (\mathbb{X}, d_{\mathbb{X}}) \rightarrow (\mathbb{Y}, d_{\mathbb{Y}})$ which characterizes the difference of f from an isometry.

Theorem 1 ([47]). 1. *Reduced to the case of metric spaces, the convergence of Definition 3 is equivalent to the Gromov-Hausdorff one;*

2) *Let (X, p) , (Y, q) be two complete pointed quasimetric spaces, each obtained as a limit of the same sequence (X_n, p_n) such that the constants $\{Q_{X_n}\}$ are uniformly bounded: $|Q_{X_n}| \leq C$ for all $n \in \mathbb{N}$. If X is boundedly compact, then X and Y are isometric.*

Remark 5. Note that a straightforward generalization of Gromov's theory to the case of quasimetric spaces is, for various reasons, impossible. For example, the Gromov-Hausdorff distance between two bounded quasimetric spaces is equal to zero [21] and, thus, makes no sense in this context. Besides that, in [25, 2] convergence is first defined for compact spaces; convergence of boundedly compact spaces is defined as convergence of all (compact) balls. For quasimetric spaces, this approach would not yield uniqueness of the limit up to isometry.

Definition 4. Let \mathbb{X} be a boundedly compact (quasi)metric space, $p \in X$. If the limit of pointed spaces $\lim_{\lambda \rightarrow \infty} (\lambda\mathbb{X}, p) = (T_p\mathbb{X}, e)$ exists (in the sense of Definition 3), then $T_p\mathbb{X}$ is called the *tangent cone* to \mathbb{X} at p . Here $\lambda\mathbb{X} = (\mathbb{X}, \lambda \cdot d_{\mathbb{X}})$; the symbol $\lim_{\lambda \rightarrow \infty} (\lambda\mathbb{X}, p)$ means

that, for any sequence $\lambda_n \rightarrow \infty$, there exists $\lim_{\lambda_n \rightarrow \infty} (\lambda_n \mathbb{X}, p)$ which is independent of the choice of sequence $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.

Any neighborhood $U(e) \subseteq T_p \mathbb{X}$ of the basepoint element $e \in T_p \mathbb{X}$ is said to be a *local tangent cone* to \mathbb{X} at p .

Remark 6. Theorem 1 implies that, for complete boundedly compact quasimetric spaces, the tangent cone is unique up to isometry, i. e. one should treat the tangent cone from Definition 4 as a class of pointed quasimetric spaces isometric to each other. Note also that the tangent cone is completely defined by any (arbitrarily small) neighborhood of the point. More precisely, if U is a neighborhood of the point $p \in \mathbb{X}$ then the tangent cones of U and \mathbb{X} at p are isometric. Moreover, the quasimetric space $(T_p \mathbb{X}, e)$ is a cone in the sense that it is invariant under scalings, i. e. $(T_p \mathbb{X}, e)$ is isometric to $(\lambda T_p \mathbb{X}, e)$ for all $\lambda > 0$.

Theorem 2 ([47]). *Let (\mathbb{X}, d, δ) be a nondegenerate dilation structure. Then $(U(x), x, d^x)$ is a local tangent cone to \mathbb{X} at x .*

Note that on the neighborhood $U(x) \subseteq \mathbb{X}$ we have two (quasi)metric structures d and d^x , thus it is natural to denote the local tangent cone to \mathbb{X} at x as $(U(x), d^x)$, not introducing any other underlying set for the tangent cone.

One of the main goals of the present paper is to describe the algebraic properties of the (local) tangent cone in the case when (\mathbb{X}, d, δ) is a strong uniform nondegenerate dilation structure. Having only axioms (A0) — (A3) we can say nothing substantial about this.

3 Algebraic properties of the tangent cone

Definition 5 ([44, 20]). A *local group* is a tuple (\mathcal{G}, e, i, p) where \mathcal{G} is a Hausdorff topological space with a fixed *identity element* $e \in \mathcal{G}$ and continuous functions $i : \Upsilon \rightarrow \mathcal{G}$ (*the inverse element function*), and $p : \Omega \rightarrow \mathcal{G}$ (*the product function*) given on some subsets $\Upsilon \subseteq \mathcal{G}$, $\Omega \subseteq \mathcal{G} \times \mathcal{G}$ such that $e \in \Upsilon$, $\{e\} \times \mathcal{G} \subseteq \Omega$, $\mathcal{G} \times \{e\} \subseteq \Omega$, and for all $x, y, z \in \mathcal{G}$ the following properties hold:

- 1) $p(e, x) = p(x, e) = x$;
- 2) if $x \in \Upsilon$, then $(x, i(x)) \in \Omega$, $(i(x), x) \in \Omega$ and $p(x, i(x)) = p(i(x), x) = e$;
- 3) if $(x, y), (y, z) \in \Omega$ and $(p(x, y), z), (x, p(y, z)) \in \Omega$, then $p(p(x, y), z) = p(x, p(y, z))$.

Assertions close to the next three propositions can be found in [6], but in our consideration, some details are different. We include the proofs for the reader's convenience.

Proposition 5. *Let (\mathbb{X}, d, δ) be a strong dilation structure. Then the function introduced in Axiom (A4) yields a product and an inverse element functions in a neighborhood of the given point. Precisely, $\mathcal{G}^x = (U(x), x, \text{inv}^x, \Sigma^x)$ (where inv^x, Σ^x are from Proposition 3) is a local group. For the inverse element, the following property holds: $\text{inv}^x(\text{inv}^x(u)) = u$.*

Proof. Let $u, v, w \in B^d(x, r)$, $\varepsilon \leq \varepsilon_0$, where r is from Axiom (A4), and ε_0 is such that $\Sigma_\varepsilon^x(u, v)$ is defined for all $\varepsilon \leq \varepsilon_0$, for example, as in Proposition 4. By direct calculation and using the uniformity of the limit in (A4) one can verify the following relations:

$$\Sigma_\varepsilon^x(x, u) = u; \quad \Sigma_\varepsilon^x(u, \delta_\varepsilon^x u) = u;$$

if both parts of the following equality are defined, then

$$\Sigma_\varepsilon^x(u, \Sigma_\varepsilon^{\delta_\varepsilon^x}(v, w)) = \Sigma_\varepsilon^x(\Sigma_\varepsilon^x(u, v), w);$$

$$\Sigma^x(u, \text{inv}_\varepsilon^x(u)) = x; \quad \Sigma^{\delta_\varepsilon^x u}(\text{inv}_\varepsilon^x(u), u) = \delta_\varepsilon^x u;$$

$$\text{inv}_\varepsilon^{\delta_\varepsilon^x u} \text{inv}_\varepsilon^x u = x.$$

Passing to the limit when $\varepsilon \rightarrow 0$, we obtain that $\Sigma^x(u, v)$ is the product function w. r. t. the identity element x and inverse function $\text{inv}^x(u)$ such that $\text{inv}^x(\text{inv}^x(u)) = u$. The domains of the product and inverse functions are some areas $\Omega \supseteq B^d(x, r) \times B^d(x, r)$, $\Upsilon \supseteq B^d(x, r)$ where r is from (A4). The continuity of functions $\Sigma^x(u, v)$ and $\text{inv}^x u$ is obvious from (A0), (A4) and Proposition 3. \square

Proposition 6. *The following identities*

$$\delta_\mu^x \Sigma^x(u, v) = \Sigma^x(\delta_\mu^x(u), \delta_\mu^x(v)), \quad \text{inv}^x(\delta_\mu^x u) = \delta_\mu^x \text{inv}^x u$$

hold provided both parts of the equality are defined (in particular, when $\Sigma^x(u, v)$ exists and $\mu \leq 1$).

Proof. For the function

$$\Lambda^x = \lim_{\varepsilon \rightarrow 0} \Lambda_\varepsilon^x(u, v) = \lim_{\varepsilon \rightarrow 0} \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v$$

from Axiom (A4), direct calculations show that $\Lambda_\varepsilon^x(\delta_\mu^x u, \delta_\mu^x v) = \delta_\mu^{\delta_\varepsilon^x u} \Lambda_{\varepsilon\mu}^x(u, v)$, hence

$$\delta_\mu^x \Lambda^x(u, v) = \Lambda^x(\delta_\mu^x u, \delta_\mu^x v), \quad (3.1)$$

provided both parts of the last equality are defined. From here the second equality of the proposition is obvious, since $\text{inv}^x(u) = \Lambda^x(u, x)$.

The first equality of the proposition follows from (3.1), (2.4) and from the second equality. \square

Proposition 7. *Let (\mathbb{X}, d, δ) be a strong nondegenerate uniform dilation structure. Then for all $u \in B^d(x, r)$ the function $\Sigma^x(u, \cdot)$ (see Proposition 3) is a d^x -isometry on $B^d(x, r)$.*

Proof. Using Proposition 2 and uniformity in Axiom (A3), we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} |d(\delta_\varepsilon^x v, \delta_\varepsilon^x w) - d^{\delta_\varepsilon^x u}(\delta_\varepsilon^x v, \delta_\varepsilon^x w)| &= \lim_{\varepsilon \rightarrow 0} | \frac{1}{\varepsilon} d(\delta_\varepsilon^x v, \delta_\varepsilon^x w) - d^{\delta_\varepsilon^x u}(\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v, \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x w) | = \\ &= |d^x(v, w) - d^x(\Lambda^x(u, v), \Lambda^x(u, w))| = 0, \end{aligned}$$

where Λ^x is from Axiom (A4). Further, we have

$$d^x(v, w) = d^x(\Lambda^x(u, \Sigma^x(u, v)), \Lambda^x(u, \Sigma^x(u, w))) = d^x(\Sigma^x(u, v), \Sigma^x(u, w)).$$

From here the assertion follows. \square

It is interesting to compare the following proposition with the definition and properties of homogeneous norm on a homogeneous Lie group [18].

Proposition 8. Let (\mathbb{X}, d, δ) be a strong nondegenerate dilation structure. Then the function $|\cdot| : B^d(x, R) \rightarrow \mathbb{R}$, defined as $|u| = d^x(x, u)$, meets the following properties:

- 1) *homogeneity*: if $u \in B^d(x, R)$ and $\delta_r^x u \in B^d(x, R)$ is defined then $|\delta_r^x u| = r|u|$;
- 2) *non-degeneracy*: $u = x$ if and only if $|u| = 0$.
- 3) *generalized triangle inequality*: if for $u, v \in B^d(x, R)$ the value $\Sigma^x(u, v) \in B^d(x, R)$ is defined then the following inequality holds:

$$|\Sigma^x(u, v)| \leq c(|u| + |v|), \quad (3.2)$$

where $1 \leq c < \infty$ and $c = c(x)$ does not depend on u, v .

Proof. The first property directly follows from the conical property; the second one is equivalent to the assumption of non-degeneracy of the dilation structure. Let us show 3). Due to continuity of the product function $(u, v) \mapsto \Sigma^x(u, v)$ there exists $0 < \tau \leq R$ such that $\bar{B}^{d^x}(x, \tau) \subseteq B^d(x, R)$ and for all $u, v \in \bar{B}^{d^x}(x, \tau)$ we have $\Sigma^x(u, v) \in B^d(x, R) \cap B^{d^x}(x, R)$. W. l. o. g. assume $|v| \leq |u|$ and consider first the case when $|u| \leq \tau$ (then $\varepsilon = \varepsilon(u) = \tau^{-1}|u| \leq 1$).

Let us show that the elements $\delta_{\tau|u|^{-1}}^x u, \delta_{\tau|u|^{-1}}^x v$ exist and belong to $B^d(x, R)$.

Indeed, it is sufficient to verify that $u \in W_{\varepsilon^{-1}}(x)$. Since $\varepsilon\tau = |u|$, we have $u \in \bar{B}^{d^x}(x, \tau\varepsilon)$ (see Remark 3). According to the choice of τ the following inclusions hold $\bar{B}^{d^x}(x, \tau) \subseteq B^d(x, R) \subseteq B^d(x, R)$, therefore, due to axiom (A0) and Proposition 2, it is true that $u \in \bar{B}^{d^x}(x, \tau\varepsilon) = \delta_\varepsilon^x \bar{B}^{d^x}(x, \tau) \subseteq \delta_\varepsilon^x B^d(x, R) \subseteq V_\varepsilon(x) \subseteq W_{\varepsilon^{-1}}(x)$. Note that it can not happen that $\delta_{\varepsilon^{-1}}^x u \in U(x) \setminus B^d(x, R)$, because $\delta_{\varepsilon^{-1}}^x B^d(x, R\varepsilon) \subseteq \delta_{\varepsilon^{-1}}^x \delta_\varepsilon^x B^d(x, R) = B^d(x, R)$.

Thus, due to 1), $|\delta_{\tau|u|^{-1}}^x u| = d^x(x, \delta_{\tau|u|^{-1}}^x u) = \tau$, $|\delta_{\tau|u|^{-1}}^x v| \leq \tau$. Hence, by choice of τ , the value $\Sigma^x(\delta_{\tau|u|^{-1}}^x u, \delta_{\tau|u|^{-1}}^x v) \in B^d(x, R) \cap B^{d^x}(x, R)$ is defined. Thus, from Proposition 6, we can derive

$$\Sigma^x(u, v) = \delta_{\tau^{-1}|u|}^x \Sigma^x(\delta_{\tau|u|^{-1}}^x u, \delta_{\tau|u|^{-1}}^x v).$$

It follows immediately that

$$\begin{aligned} |\Sigma^x(u, v)| &= |\delta_{\tau^{-1}|u|}^x (\Sigma^x(\delta_{\tau|u|^{-1}}^x u, \delta_{\tau|u|^{-1}}^x v))| \\ &= \tau^{-1}|u| |\Sigma^x(\delta_{\tau|u|^{-1}}^x u, \delta_{\tau|u|^{-1}}^x v)| \leq c|u| \leq c(|u| + |v|), \end{aligned}$$

where $c = \tau^{-1}R$.

Let now be $|u| > \tau$ and $\Sigma^x(u, v) \in B^d(x, R)$ be defined. Choose $0 < \mu < 1$ such that $\delta_\mu^x u, \delta_\mu^x v \in B^{d^x}(x, \tau)$ (such μ exists due to continuity of dilations). Then

$$\mu |\Sigma^x(u, v)| = |\delta_\mu^x \Sigma^x(u, v)| = |\Sigma^x(\delta_\mu^x u, \delta_\mu^x v)| \leq c(|\delta_\mu^x u| + |\delta_\mu^x v|) = c\mu(|u| + |v|).$$

It follows (3.2). □

Definition 6. The function $|\cdot|$, introduced in Proposition 8, is said to be *the homogeneous norm* on the local group \mathcal{G}^x .

Definition 7 ([31]). It is said that for a local group \mathcal{G} the *global associativity property* holds if there is a neighborhood of the identity $V \subseteq \mathcal{G}$ such that for each n -tuple of elements $a_1, a_2, \dots, a_n \in V$ whenever there exist two different ways of introducing parentheses in this n -tuple, so that all intermediate products are defined, the resulting products are equal.

Theorem 3 (Mal'tsev [31]). *A local topological group \mathcal{G} is locally isomorphic to a some topological group G if and only if the global associativity property in \mathcal{G} holds.*

Remark 7. Unlike in the case of global groups, the verification of the global associativity property for local groups is a nontrivial task. This verification can not be done by a trivial induction as for global groups since it would require to assume the existence of all intermediate products which is, in general, not true for local groups. See comments in [40, 20] where there are some references to papers with mistakes caused by misunderstandings of this fact. In the local group \mathcal{G}^x under our consideration it is easy to provide examples for $n = 4$ such that $u_i \in B^d(x, R)$ and combinations $u = \Sigma^x(\Sigma^x(u_1, \Sigma^x(u_2, u_3)), u_4)$ and $u' = \Sigma^x(u_1, \Sigma^x(u_2, \Sigma^x(u_3, u_4)))$ exist while the combination $\Sigma^x(\Sigma^x(u_1, u_2), \Sigma^x(u_3, u_4))$ is not defined. More examples can be found in [31, 40].

Proposition 9. *For the local group \mathcal{G}^x , the global associativity property holds.*

Proof. Let $u_1, u_2, \dots, u_n \in B^d(x, R)$, and u, u' be elements obtained from the n -tuple (u_1, u_2, \dots, u_n) by introducing parentheses such that the products exist. We need to show that $u = u'$.

Let τ be such as in the proof of Proposition 8, $R_x = \inf\{\xi \mid B^d(x, R) \subseteq B^{d^x}(x, \xi)\}$, $c_n = nc^{n-1}$ where c is from (3.2). Let $s_n = \frac{\tau}{c_{n-1}R_x}$ and $\tilde{u}_i = \delta_{s_n}^x u_i$. By induction on n and using (3.2) it is easy to show that all possible products of length not bigger than n of the elements \tilde{u}_i are defined. Thus it can be trivially shown (as for global groups) that $\delta_{s_n}^x(u) = \delta_{s_n}^x(u')$. Applying to both sides of the last equality the homeomorphism $\delta_{s_{n-1}}^{x_{n-1}}$ (which is, in particular, an injective mapping), we get $u = u'$ and finish the proof. \square

Definition 8 ([48], Proposition 5.4). A topological group G is *contractible* if there is an automorphism $\tau : G \rightarrow G$ such that $\lim_{n \rightarrow \infty} \tau^n g = e$ for all $g \in G$.

Definition 9. A topological space is *locally compact* if any of its points has a neighborhood the closure of which is compact. A local group is *locally compact* if there is a neighborhood of its identity element the closure of which is compact.

The proof of Theorem 4 relies on the following statement, see Remark 2 for comments.

Proposition 10 ([48], Corollary 2.4). *For a connected locally compact group G the following assertions are equivalent:*

- (1) G is contractible;
- (2) G is a simply connected Lie group the Lie algebra V of which is nilpotent and graded, i. e. there is a decomposition $V = \bigoplus_{s>0} V_s$ such that $[V_s, V_t] \subseteq V_{s+t}$ for all $s, t > 0$. In particular, V is nilpotent.

Theorem 4. *Let (\mathbb{X}, d, δ) be a strong nondegenerate dilation structure. Then*

- 1) *For any $x \in \mathbb{X}$, the local group \mathcal{G}^x is locally isomorphic to a connected simply connected Lie group G^x the Lie algebra of which is nilpotent and graded;*
- 2) *If the dilation structure is, in addition, uniform, then the Lie group G^x is the tangent cone (in the sense of Definition 4) to \mathbb{X} at x , i. e., left translations on G^x are isometries w. r. t. quasimetric \tilde{d}^x on G^x which arises from d^x in a natural way. The local group \mathcal{G}^x is a local tangent cone.*

Proof. Since \mathbb{X} is boundedly compact, \mathcal{G}^x is a locally compact local group. Due to existence on \mathcal{G}^x of a one-parameter family of dilations this local group is linearly connected (indeed, any two points $u, v \in U(x)$ can be connected by the continuous curve $\{\delta_\varepsilon^x(u)\}_{1 \geq \varepsilon \geq 0} \circ \{\delta_\varepsilon^x(v)\}_{0 \leq \varepsilon \leq 1}$), hence \mathcal{G}^x is connected.

According to Proposition 9, the global associativity property in \mathcal{G}^x holds. Hence, by Theorem 3, \mathcal{G}^x is locally isomorphic to some topological group G^x . Let us use the construction of this group given in the proof of Theorem 3 in [31] and in more details in [16]: G^x is obtained as the group of equivalence classes of words arranged from elements of the initial local group \mathcal{G}^x .

Namely, let $\mathcal{G}_{(n)}^x = \{(u_1, \dots, u_n) \mid u_i \in \mathcal{G}^x\}$ be the set of words of length n , and $\tilde{G}^x = \bigcup_{n \in \mathbb{N}} \mathcal{G}_{(n)}^x$. On \tilde{G}^x the following two operations can be introduced. The contraction is defined as

$$(u_1, \dots, u_n) \in \mathcal{G}_{(n)}^x \mapsto (u_1, \dots, u_{i-1}, \Sigma^x(u_i, u_{i+1}), u_{i+2}, \dots, u_n) \in \mathcal{G}_{(n-1)}^x,$$

if $\Sigma^x(u_i, u_{i+1})$ exists. The expansion is defined as

$$(u_1, \dots, u_n) \in \mathcal{G}_{(n)}^x \mapsto (u_1, \dots, u_{i-1}, v, w, u_{i+1}, \dots, u_n) \in \mathcal{G}_{(n+1)}^x,$$

if $u_i = \Sigma^x(v, w)$. Two words (u_1, \dots, u_n) and (v_1, \dots, v_m) are called equivalent (which is denoted as $(u_1, \dots, u_n) \sim (v_1, \dots, v_m)$) if they can be obtained one from another by a finite sequence of contractions and expansions. Finally, let $G^x = \tilde{G}^x / \sim$. The product and inverse functions and the neutral element on G^x are defined respectively as

$$\begin{aligned} [(u_1, \dots, u_n)] \cdot [(v_1, \dots, v_m)] &= [(u_1, \dots, u_n, v_1, \dots, v_m)], \\ [(u_1, \dots, u_n)]^{-1} &= [(\text{inv}^x u_n, \dots, \text{inv}^x u_1)], \quad e_{G^x} = [(e_{\mathcal{G}^x})]. \end{aligned}$$

It is easy to verify that the function $\varphi : \mathcal{G}^x \rightarrow G^x$ which maps the element g to the equivalence class $[(g)]$, is a local isomorphism.

The topology on G^x is defined as follows: if \mathcal{B} is the basis of topology of \mathcal{G}^x , then $B = \{\varphi(U) \mid U \in \mathcal{B}\}$ is the base of topology of G^x . The verification of axioms of a topological basis can be done straightforwardly.

For an arbitrary $s < 1$ define a contractive automorphism on G^x as

$$\tau([(u_1, \dots, u_n)]) = [(\delta_s^x(u_1), \dots, \delta_s^x(u_n))].$$

Due to the linear connectedness of the group G^x (because of the obvious relation $[(e, e, \dots, e)] = [(e)]$ and the fact that the local group \mathcal{G}^x is linearly connected), by Proposition 10 we get the first assertion of the theorem.

Now let $s_{mn} = s_{\max\{m, n\}}$ (in notation of the proof of Proposition 9) and define on G^x a quasimetric as

$$\begin{aligned} \tilde{d}^x([(u_1, \dots, u_n)], [(v_1, \dots, v_m)]) \\ = \frac{1}{s_{mn}} d^x(\Sigma^x(\delta_{s_{mn}}^x u_1, \dots, \delta_{s_{mn}}^x u_n), \Sigma^x(\delta_{s_{mn}}^x v_1, \dots, \delta_{s_{mn}}^x v_m)). \end{aligned}$$

Note that Propositions 2, 6 imply the generalized triangle inequality for \tilde{d}^x with the constant $Q_{\mathbb{X}}$ and that φ is an isometry. Taking into account Theorem 2 and Proposition 7 we obtain the second assertion. \square

Remark 8. Let us give a brief overview of the proof of Proposition 10, for showing that it can not be straightforwardly applied to the case of local groups. The crucial part of this proof is to show that a connected locally compact contractible group is a Lie group. This proof heavily relies on several main theorems from the book of Montgomery and Zippin [37], where the solution of H5 is given. The proofs of those theorems are long and complicated, and, as noted in [37, p. 119], “Most of the Lemmas can be also proved by essentially the same arguments for the case of a locally compact connected *local* group but we shall not complicate the statements and proofs of the Lemmas by inserting the necessary qualifications.” This last statement shows, that proving the theorems (based on the mentioned lemmas) that we would need, for the case of local groups, is, at least, nontrivial (and not done by anybody, as far as we know). It would require a careful study of large parts of the book [37].

Overcoming this difficulty we apply Mal’tsev’s theorem 3 to reduce the consideration to the case of (global) groups, for which Proposition 10 can be applied.

Remark 9. There is another look at the proof of Proposition 9. It actually can be proved without the triangle inequality (3.2) and any (quasi)metric structure, by means of the following simple topological fact ([44, Chapter 3, Section 23, E], see also [20]): in any local group there is a decreasing sequence of neighborhoods $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ such that, for all elements $u_1, \dots, u_n \in \mathcal{U}_n$, their products are defined with any combinations of parentheses. Using this fact, an analog of Theorem 4, for locally compact topological spaces with dilations, can be proved (for this purpose, axioms of Definition 2 should be modified in a natural way). Globalizability of locally compact locally connected contractible local groups was proved in [16], independently of our paper. The result of [16] can be viewed as a generalization of the first assertion of Theorem 4.

On the other hand, using the (quasi)metric structure allows to make the proof of global associativity more constructive in comparison with the purely topological one.

4 Example: Carnot-Carathéodory spaces

Definition 10 ([2, 25, 28, 39, 29, 52, 53]). Fix a connected Riemannian C^∞ -manifold \mathbb{M} of dimension N . The manifold \mathbb{M} is called a *regular Carnot-Carathéodory space* if in the tangent bundle $T\mathbb{M}$ there is a filtration

$$H\mathbb{M} = H_1\mathbb{M} \subseteq \dots \subseteq H_i\mathbb{M} \subseteq \dots \subseteq H_M\mathbb{M} = T\mathbb{M}$$

of subbundles of the tangent bundle $T\mathbb{M}$, such that, for each point $p \in \mathbb{M}$, there exists a neighborhood $U \subset \mathbb{M}$ with a collection of $C^{1,\alpha}$ (where $\alpha \in (0, 1]$) vector fields X_1, \dots, X_N on U enjoying the following three properties. For each $v \in U$ we have

- (1) $X_1(v), \dots, X_N(v)$ constitutes a basis of $T_v\mathbb{M}$;
- (2) $H_i(v) = \text{span}\{X_1(v), \dots, X_{\dim H_i}(v)\}$ is a subspace of $T_v\mathbb{M}$ of dimension $\dim H_i$, $i = 1, \dots, M$, where $H_1(v) = H_v\mathbb{M}$;
- (3)

$$[X_i, X_j](v) = \sum_{\deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v) \quad (4.1)$$

where the *degree* $\deg X_k$ equals $\min\{m \mid X_k \in H_m\}$;

The number M is called the *depth* of the Carnot-Carathéodory space.

Remark 10. According to [29], all statements below are also valid for the case when $X_i \in C^1$ and $M = 2$.

Definition 11. For any point $g \in \mathbb{M}$, define the mapping

$$\theta_g(v_1, \dots, v_N) = \exp\left(\sum_{i=1}^N v_i X_i\right)(g). \quad (4.2)$$

It is known that θ_g is a C^1 -diffeomorphism of the Euclidean ball $B_E(0, r) \subseteq \mathbb{R}^N$ to \mathbb{M} , where $0 \leq r < r_g$ for some (small enough) r_g . The collection $\{v_i\}_{i=1}^N$ is called *the normal coordinates or the coordinates of the 1st kind (with respect to $u \in \mathbb{M}$)* of the point $v \in U_g = \theta_g(B_E(0, r_g))$. Further we will consider a compactly embedded neighborhood $\mathcal{U} \subseteq \mathbb{M}$ such that $\mathcal{U} \subseteq \bigcap_{g \in \mathcal{U}} U_g$.

Definition 12. By means of coordinates (4.2), introduce on \mathcal{U} the following quasimetric d_∞ . For $u, v \in \mathcal{U}$ such that $v = \exp\left(\sum_{i=1}^N v_i X_i\right)(u)$ let

$$d_\infty(u, v) = \max_i \{|v_i|^{\frac{1}{\deg X_i}}\}.$$

The properties (1), (2) of Definition 1 for the function d_∞ and its continuity on both arguments obviously follow from properties of the exponential mapping. The generalized triangle inequality is proved in [28, 29]. We denote the balls w. r. t. d_∞ as $\text{Box}(u, r) = \{v \in \mathcal{U} \mid d_\infty(v, u) < r\}$.

Definition 13. Define in \mathcal{U} the action of the dilation group Δ_ε^g as follows: it maps an element $x = \exp\left(\sum_{i=1}^N x_i X_i\right)(g) \in \mathcal{U}$ to the element

$$\Delta_\varepsilon^g x = \exp\left(\sum_{i=1}^N x_i \varepsilon^{\deg X_i} X_i\right)(g) \in \mathcal{U}$$

in the case when the right-hand part of the last expression makes sense.

Proposition 11 ([29]). *The coefficients*

$$\bar{c}_{ijk} = \begin{cases} c_{ijk}(g) \text{ of (4.1) ,} & \text{if } \deg X_i + \deg X_j = \deg X_k \\ 0, & \text{in other cases} \end{cases}$$

define a graded nilpotent Lie algebra.

This Lie algebra can be represented by vector fields $\{(\hat{X}_i^g)\}_{i=1}^N \in C^\alpha$ on \mathcal{U} such that

$$[\hat{X}_i^g, \hat{X}_j^g] = \sum_{\deg X_k = \deg X_i + \deg X_j} c_{ijk}(g) \hat{X}_k^g \quad (4.3)$$

and $\hat{X}_i^g(g) = X_i(g)$.

Definition 14. To the Lie algebra $\{\widehat{X}_i^g\}_{i=1}^N$ there corresponds the Lie group $\mathcal{G}^g = (\mathcal{U}, g, {}^{-1}, *)$ at g . The product function $*$ is defined as follows: if $x = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^g\right)(g)$, $y = \exp\left(\sum_{i=1}^N y_i \widehat{X}_i^g\right)(g)$, then $x*y = \exp\left(\sum_{i=1}^N y_i \widehat{X}_i^g\right) \circ \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^g\right)(g) = \exp\left(\sum_{i=1}^N z_i \widehat{X}_i^g\right)(g)$, where z_i are computed via Campbell-Hausdorff formula. The inverse element to $x = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^g\right)(g)$ is defined as $x^{-1} = \exp\left(\sum_{i=1}^N (-x_i) \widehat{X}_i^g\right)(g)$.

Remark 11. In the “classical” sub-Riemannian setting (see Introduction), the local Lie group from Definition 14 is locally isomorphic to a *Carnot group*, i.e., a connected simply connected Lie group the Lie algebra V of which can be decomposed into a direct sum $V = V_1 \oplus \dots \oplus V_M$ such that $[V_1, V_i] = V_{i+1}$, $i = 1, \dots, M-1$, $[V_1, V_M] = \{0\}$. In the case under our consideration, for the Lie algebra of the local group \mathcal{G}^g only the inclusion $[V_1, V_i] \subseteq V_{i+1}$ is true. The converse inclusion will hold if we require an additional condition [28, 29] in Definition 10: the quotient mapping $[\cdot, \cdot]_0 : H_1 \times H_j / H_{j-1} \mapsto H_{j+1} / H_j$ induced by Lie brackets is an epimorphism for all $1 \leq j < M$. Under this additional assumption, an analog of the Rashevskii-Chow theorem can be proved.

Strictly speaking, the group operation is defined on a neighborhood defined by vector fields $\{\widehat{X}_i^g\}$, but, w. l. o. g., we can assume that this neighborhood coincides with \mathcal{U} [29, 53]. Note also that the mapping θ_g is a local isometric isomorphism between the local Lie group $(\mathcal{G}^g, *)$ and the Lie group $(\mathbb{R}^N, *)$, and $\theta_g(0) = g$. The group operation $*$ on \mathbb{R}^N is introduced by analogy with Definition 14, by means of C^∞ vector fields $\{(\widehat{X}_i^g)'\}$ on \mathbb{R}^N , such that $\widehat{X}_i^g = (\theta_g)_*(\widehat{X}_i^g)'$, where $(\theta_g)_*(Y)(\theta_g(x)) = D\theta_g(x)\langle Y(x) \rangle$, $Y \in T\mathbb{R}^N$ (see details in [28, 29, 53]). In what follows, we will identify the neighborhood \mathcal{U} with its image $\theta_g^{-1}(\mathcal{U}) \subseteq \mathbb{R}^N$.

This identification allows, in particular, to define canonical coordinates of the first kind, induced by the nilpotentized vector fields in a similar way as 11.

Definition 15. For $u, v \in \mathbb{R}^N$ such that $v = \exp\left(\sum_{i=1}^N v_i (\widehat{X}_i^g)'\right)(u)$, let $d_\infty^g(u, v) = \max_i \{|v_i|^{\frac{1}{\deg X_i}}\}$.

It is known [18] that d_∞^g is a quasimetric. We denote the balls w. r. t. this quasimetric as $\text{Box}^g(u, r) = \{v \in \mathbb{R}^N \mid d_\infty^g(v, u) < r\}$.

Proposition 12 ([29, 50]). *If r is such that $\text{Box}(g, r) \subseteq \mathcal{U}$ then $\text{Box}(g, r) = \text{Box}^g(g, r)$.*

Definition 16. The nilpotentized vector fields also define dilations on \mathcal{U} : the element $x = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^g\right)(g) \in \mathcal{U}$ is mapped to the element

$$\delta_{g,\varepsilon}^g x = \exp\left(\sum_{i=1}^N x_i \varepsilon^{\deg X_i} \widehat{X}_i^g\right)(g) \in \mathcal{U}$$

in the case when the right-hand part of the last expression makes sense.

Proposition 13 ([29, 50]). *For all $\varepsilon > 0$ and $u \in \mathcal{U}$, we have $\Delta_\varepsilon^g u = \delta_{g,\varepsilon}^g u$, if both parts of this equality are defined.*

Proposition 14 ([18, 29, 53]). *The cone property for the quasimetric $d_\infty^g(u, v)$ holds: $d_\infty^g(u, v) = \frac{1}{\varepsilon} d_\infty^g(\Delta_\varepsilon^g u, \Delta_\varepsilon^g v)$ for all possible $\varepsilon > 0$.*

Theorem 5 (Estimate on divergence of integral lines [28, 29]). *Consider points $u, v \in \mathcal{U}$ and*

$$w_\varepsilon = \exp\left(\sum_{i=1}^N w_i \varepsilon^{\deg X_i} X_i\right)(v) \text{ and } \widehat{w}_\varepsilon = \exp\left(\sum_{i=1}^N w_i \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(v).$$

Then

$$\max\{d_\infty^u(w_\varepsilon, \widehat{w}_\varepsilon), d_\infty^{u'}(w_\varepsilon, \widehat{w}_\varepsilon)\} = \varepsilon[\Theta(u, v, \alpha, M)]\rho(u, v)^{\frac{\alpha}{M}}, \quad (4.4)$$

where Θ is uniformly bounded on $u, v \in \mathcal{U}$.

Theorem 6 (Local approximation theorem [2, 21, 25, 28, 29, 52]). *If $u, v \in \text{Box}(g, \varepsilon)$, then $|d_\infty(u, v) - d_\infty^g(u, v)| = O(\varepsilon^{1+\frac{\alpha}{M}})$ uniformly on $g \in \mathcal{U}$, $u, v \in \text{Box}(g, \varepsilon)$.*

Theorem 7. *Dilations from Definition 13 induce on the quasimetric space (\mathcal{U}, d_∞) a strong uniform nondegenerate dilation structure with the conical quasimetric (d^x from Axiom (A3)) d_∞^g .*

Proof. Axioms (A0) — (A2) and non-degeneracy of Definition 2 obviously hold due to properties of exponential mappings; (A3) and uniformity directly follow from Theorem 6.

Axiom (A4) follows from group operation properties and Theorem 5. Indeed, let $u = \exp\left(\sum_{i=1}^N u_i X_i\right)(g)$, $v = \exp\left(\sum_{i=1}^N v_i X_i\right)(g) \in \mathcal{U}$. We need to show the existence and uniformity of the limits of $\Sigma_\varepsilon^g(u, v) = \Delta_{\varepsilon^{-1}}^g \Delta_\varepsilon^g u v$ and $\text{inv}_\varepsilon^g(u) = \Delta_{\varepsilon^{-1}}^g u g$, when $\varepsilon \rightarrow 0$ (see Proposition 3).

First we prove the existence of the limit on the local group (i. e. replacing Δ_ε^g by $\delta_{g, \varepsilon}^g$). According to (A2), $\lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon^g u = \lim_{\varepsilon \rightarrow 0} u_\varepsilon = g$. By means of (11) we can write

$$v = \exp\left(\sum_{i=1}^N \tilde{v}_i^\varepsilon X_i\right)(u_\varepsilon).$$

Since the coordinates of the first kind are uniquely defined,

$$\lim_{\varepsilon \rightarrow 0} \tilde{v}_i^\varepsilon = v_i, \quad i = 1, \dots, N. \quad (4.5)$$

Now let

$$a = \delta_{g, \varepsilon}^{u_\varepsilon} v = \exp\left(\sum_{i=1}^N \tilde{v}_i^\varepsilon \varepsilon^{\deg X_i^g} \widehat{X}_i^g\right) \circ \exp\left(\sum_{i=1}^N u_i \varepsilon^{\deg X_i} \widehat{X}_i^g\right)(g).$$

Then

$$\Sigma_\varepsilon^g(u, v) = \delta_{g, \varepsilon^{-1}}^g a = \exp\left(\sum_{i=1}^N \tilde{v}_i^\varepsilon (\delta_{g, \varepsilon^{-1}}^g)_*(\varepsilon^{\deg X_i} \widehat{X}_i^g)\right) \circ \exp\left(\sum_{i=1}^N u_i (\delta_{g, \varepsilon^{-1}}^g)_*(\varepsilon^{\deg X_i} \widehat{X}_i^g)\right)(g).$$

Using group homogeneity and (4.5), we get the existence of the uniform (on g) limit

$$\lim_{\varepsilon \rightarrow 0} \Sigma_\varepsilon^g(u, v) = \exp\left(\sum_{i=1}^N v_i \widehat{X}_i^g\right) \circ \exp\left(\sum_{i=1}^N u_i \widehat{X}_i^g\right)(g).$$

Now let us estimate the difference between the two combinations. From Properties 13, 14 and Theorem 5 we infer

$$\begin{aligned} d_\infty^g \left(\Delta_{\varepsilon^{-1}}^g \Delta_\varepsilon^{\Delta_\varepsilon^g u} v, \delta_{g, \varepsilon^{-1}}^g \delta_{g, \varepsilon}^{\Delta_\varepsilon^g u} v \right) &= d_\infty^g \left(\Delta_{\varepsilon^{-1}}^g \Delta_\varepsilon^{\Delta_\varepsilon^g u} v, \Delta_{\varepsilon^{-1}}^g \delta_{g, \varepsilon}^{\Delta_\varepsilon^g u} v \right) = \\ &= \varepsilon^{-1} d_\infty^g \left(\Delta_\varepsilon^{u_\varepsilon} v, \delta_{g, \varepsilon}^{u_\varepsilon} v \right) = \varepsilon^{-1} \cdot O \left(\varepsilon^{1+\frac{1}{\alpha}} \right) \rightarrow 0 \end{aligned}$$

when $\varepsilon \rightarrow 0$, which implies the uniform convergence of $\Sigma_\varepsilon^g(u, v)$.

Concerning the inverse element, we have

$$u_\varepsilon = \exp \left(\sum_{i=1}^N u_i \varepsilon^{\deg X_i} X_i \right) (g), \quad g = \exp \left(\sum_{i=1}^N -u_i \varepsilon^{\deg X_i} X_i \right) (u_\varepsilon),$$

hence

$$\text{inv}^g(u, v) = \text{inv}_\varepsilon^g(u, v) = \exp \left(\sum_{i=1}^N -u_i X_i \right) (g),$$

which finishes the proof. □

Remark 12. In contrast to the proof of a similar assertion in [8], we do not use, for proving Theorem 7, the normal frames technique [2].

Nevertheless, our considerations include, as a particular case, the “classical” sub-Riemannian setting, although in this setting the number of nontrivial commutators of “horizontal” vector fields can be bigger than the dimension N of the manifold \mathbb{M} . Indeed, the nilpotent Lie algebras, defined by different bases, are isomorphic to each other due to the functorial property of the tangent cone [50, 29]. Analogs of the basic Theorems 6, 5, needed for the proof of Theorem 7 for the intrinsic metric d_c are proved in [2, 29, 52].

Remark 13. An analog of Theorem 7 can be proved for some other quasimetrics equivalent to d_∞ , looking like e. g. in [2].

Note also that proofs in [28] do not use tools concerned with the Baker-Campbell-Hausdorff formula.

5 Differentiability

Let $(\mathbb{X}, d_\mathbb{X}, \delta)$ and $(\mathbb{Y}, d_\mathbb{Y}, \tilde{\delta})$ be two quasimetric spaces with strong nondegenerate dilation structures. In this section we denote the local group \mathcal{G}^x at $x \in \mathbb{X}$ (\mathcal{G}^y at $y \in \mathbb{Y}$) by the symbol $\mathcal{G}^x \mathbb{X}$ ($\mathcal{G}^y \mathbb{Y}$). Quasimetrics on them will be denoted by d^x and d^y respectively.

Recall that a δ -homogeneous homomorphism of graded nilpotent groups \mathbb{G} and $\tilde{\mathbb{G}}$ with one-parameter groups of dilations δ and $\tilde{\delta}$ [18] respectively is a continuous homomorphism $L : \mathbb{G} \rightarrow \tilde{\mathbb{G}}$ of these groups such that

$$L \circ \delta = \tilde{\delta} \circ L.$$

The case of local graded nilpotent groups \mathcal{G} and $\tilde{\mathcal{G}}$ with one-parameter groups of dilations δ and $\tilde{\delta}$ respectively is different from this only in that the equality $L \circ \delta(v) = \tilde{\delta} \circ L(v)$ holds only for $v \in \mathcal{G}$ and $t > 0$ such that $\delta_t v \in \mathcal{G}$ and $\tilde{\delta}_t L(v) \in \tilde{\mathcal{G}}$.

Definition 17. Given two quasimetric spaces $(\mathbb{X}, d_{\mathbb{X}}, \delta)$ and $(\mathbb{Y}, d_{\mathbb{Y}}, \tilde{\delta})$ with strong uniform nondegenerate dilation structures, and a set $E \subset \mathbb{X}$. A mapping $f : E \rightarrow \mathbb{Y}$ is called δ -differentiable at a point $g \in E$ if there exists a δ -homogeneous homomorphism $L : (\mathcal{G}^g \mathbb{X}, d^g) \rightarrow (\mathcal{G}^{f(g)} \mathbb{Y}, d^{f(g)})$ of the local nilpotent tangent cones such that

$$d^{f(g)}(f(v), L(v)) = o(d^g(g, v)) \quad \text{as } E \cap \mathcal{G}^g \mathbb{X} \ni v \rightarrow g. \quad (5.1)$$

A δ -homogeneous homomorphism $L : (\mathcal{G}^g \mathbb{X}, d^g) \rightarrow (\mathcal{G}^{f(g)} \mathbb{Y}, d^{f(g)})$ satisfying condition (5.1), is called a δ -differential of the mapping $f : E \rightarrow \mathbb{Y}$ at $g \in E$ on E and is denoted by $Df(g)$. It can be proved like in [50, 51] that if $E = \mathbb{X}$ then the δ -differential is unique.

Moreover, it is easy to verify that a homomorphism $L : (\mathcal{G}^g \mathbb{X}, d^g) \rightarrow (\mathcal{G}^{f(g)} \mathbb{Y}, d^{f(g)})$ satisfying (5.1) commutes with the one-parameter dilation group:

$$\tilde{\delta}_t^{f(g)} \circ L = L \circ \delta_t^g, \quad (5.2)$$

i.e., L is δ -homogeneous homomorphism.

In the case of Carnot groups, the introduced concept of differentiability coincides with the concept of P -differentiability given by P. Pansu in [42].

The following assertion is similar to the corresponding statement of [51, Proposition 2.3].

Proposition 15. *Definition 17 is equivalent to each of the following assertions:*

- 1) $d^{f(g)}(\tilde{\delta}_{t^{-1}}^{f(g)} f(\delta_t^g(v)), L(v)) = o(1)$ as $t \rightarrow 0$, where $o(\cdot)$ is uniform in the points v of any compact part of $\mathcal{G}^g \mathbb{X}$;
- 2) $d^{f(g)}(f(v), L(v)) = o(d_{\mathbb{X}}(g, v))$ as $E \cap \mathcal{G}^g \mathbb{X} \ni v \rightarrow g$;
- 3) $d_{\mathbb{Y}}(f(v), L(v)) = o(d^g(g, v))$ as $E \cap \mathcal{G}^g \mathbb{X} \ni v \rightarrow g$;
- 4) $d_{\mathbb{Y}}(f(v), L(v)) = o(d_{\mathbb{X}}(g, v))$ as $E \cap \mathcal{G}^g \mathbb{X} \ni v \rightarrow g$;
- 5) $d_{\mathbb{Y}}(f(\delta_t^g(v)), L(\delta_t^g v)) = o(t)$ as $t \rightarrow 0$, where $o(\cdot)$ is uniform in the points v of any compact part of $\mathcal{G}^g \mathbb{X}$.

Proof. Consider a point v of a compact part of $\mathcal{G}^g \mathbb{X}$ and a sequence $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$ such that $\delta_{\varepsilon_i}^g v \in E$ for all $i \in \mathbb{N}$. From (5.1) we have $d^{f(g)}(f(\delta_{\varepsilon_i}^g v), L(\delta_{\varepsilon_i}^g v)) = o(d^g(g, \delta_{\varepsilon_i}^g v)) = o(\varepsilon_i)$. In view of (5.2), we infer

$$d^{f(g)}(\tilde{\delta}_{\varepsilon_i}^{f(g)}(\tilde{\delta}_{\varepsilon_i^{-1}}^{f(g)} f(\delta_{\varepsilon_i}^g v)), \tilde{\delta}_{\varepsilon_i}^{f(g)} L(v)) = o(\varepsilon_i) \quad \text{uniformly in } v.$$

From here, applying the cone property of Proposition 2, we obtain just item 1. Obviously, the arguments are reversible. Item 1 is equivalent to item 5 since in view of (2.1) we have

$$\begin{aligned} & |d_{\mathbb{Y}}(\tilde{\delta}_{\varepsilon_i}^{f(g)}(\tilde{\delta}_{\varepsilon_i^{-1}}^{f(g)} f(\delta_{\varepsilon_i}^g v)), \tilde{\delta}_{\varepsilon_i}^{f(g)} L(v)) - d^{f(g)}(\tilde{\delta}_{\varepsilon_i}^{f(g)}(\tilde{\delta}_{\varepsilon_i^{-1}}^{f(g)} f(\delta_{\varepsilon_i}^g v)), \tilde{\delta}_{\varepsilon_i}^{f(g)} L(v))| \\ &= |d_{\mathbb{Y}}(\tilde{\delta}_{\varepsilon_i}^{f(g)}(\tilde{\delta}_{\varepsilon_i^{-1}}^{f(g)} f(\delta_{\varepsilon_i}^g v)), \tilde{\delta}_{\varepsilon_i}^{f(g)} L(v)) - o(\varepsilon_i)| = o(\varepsilon_i) \quad \text{uniformly in } v. \end{aligned} \quad (5.3)$$

Item 5 implies item 3 and vice versa. By comparing the metrics: $d^g(g, v) = O(d_{\mathbb{X}}(g, v))$ and $d_{\mathbb{X}}(g, v) = O(d^g(g, v))$, we obtain the equivalence of the items 3 and 4. The proof of an equivalence of the items 4 and 2 is similar to (5.3). \square

Let us generalize the chain rule of paper [51].

Theorem 8. Suppose that $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ are three quasimetric spaces with strong uniform non-degenerate dilation structures, E is a set in \mathbb{X} , and $f : E \rightarrow \mathbb{Y}$ is a mapping from E into \mathbb{Y} δ -differentiable at a point $g \in E$. Suppose also that F is a set in \mathbb{Y} , $f(E) \subset F$ and $\varphi : F \rightarrow \mathbb{Z}$ is a mapping from F into \mathbb{Z} δ -differentiable at $p = f(g) \in F$. Then the composition $\varphi \circ f : E \rightarrow \mathbb{Z}$ is δ -differentiable at g and

$$D(\varphi \circ f)(g) = D\varphi(p) \circ Df(g).$$

Proof. By hypothesis, $d^{f(g)}(f(v), Df(g)(v)) = o(d^g(g, v))$ as $v \rightarrow g$ and also $d^{\varphi(p)}(\varphi(w), D\varphi(p)(w)) = o(d^p(p, w))$ as $w \rightarrow p$. It follows that f is continuous in $g \in E$ and φ is continuous in $p \in F$. We now infer

$$\begin{aligned} d^{\varphi(p)}((\varphi \circ f)(v), (D\varphi(p) \circ Df(g))(v)) \\ \leq Q_{\mathbb{Z}}[d^{\varphi(p)}(\varphi(f(v)), D\varphi(p)(f(v))) + d^{\varphi(p)}(D\varphi(p)(f(v)), D\varphi(p)(Df(g)(v)))] \\ \leq o(d^p(p, f(v))) + O(d^p(f(v), Df(g)(v))) \\ \leq o(d^g(g, v)) + O(o(d^g(g, v))) = o(d^g(g, v)) \quad \text{as } v \rightarrow g, \end{aligned}$$

since

$$\begin{aligned} d^p(p, f(v)) &\leq Q_{\mathbb{Y}}[d^p(p, Df(g)(v)) + d^p(f(v), Df(g)(v))] \\ &= O(d^g(g, v)) + o(d^g(g, v)) = O(d^g(g, v)) \quad \text{as } v \rightarrow g. \end{aligned}$$

(The estimate $d^p(p, Df(g)(v)) = O(d^g(g, v))$ as $v \rightarrow g$ follows from the continuity of the homomorphism $Df(g)$ and (5.2).) \square

Remark 14. Note that the concept of differentiability for the quasiconformal mappings of Carnot-Carathéodory manifolds was first suggested by Margulis and Mostow in [32] and is essentially based on Mitchell's paper [34]: *A quasiconformal mapping $\varphi : \mathbf{M} \rightarrow \mathbf{N}$ is differentiable at a point x_0 in the sense of [32] if the family of mappings $\varphi_t : (\mathbf{M}, td_{\mathbf{M}}) \rightarrow (\mathbf{N}, td_{\mathbf{N}})$ induced by the mapping $\varphi : (\mathbf{M}, d_{\mathbf{M}}) \rightarrow (\mathbf{N}, d_{\mathbf{N}})$ converges to a horizontal homomorphism of the tangent cones at the points $x_0 \in \mathbf{M}$ and $\varphi(x_0) \in \mathbf{N}$ as $t \rightarrow \infty$ uniformly on compact sets.* Unfortunately, this definition is not well suitable for studying the differentials. The problem is that the tangent cone is a class of isometric spaces. Dealing with differentials, one would prefer to know what happens in a fixed direction of a tangent space. In this context, in applications of differentials it is important to know how a concrete representative of the tangent cone is geometrically and analytically connected with the given (quasi)metric space.

References

- [1] Agrachev A.A., Sachkov Yu.L. Control theory from the geometric viewpoint. 2004.
- [2] Bellaïche A. The tangent space in sub-Riemannian geometry. Sub-Riemannian Geometry, Progress in Mathematics, 144. Birkhäuser, 1996. pp. 1–78.
- [3] Berestovskii V. N. Homogeneous manifolds with an intrinsic metric. I. Sibirsk. Mat. Zh. **29** (6) (1988) 17–29.

- [4] Bramanti M., Brandolini L., Pedroni M. Basic properties of nonsmooth Hörmander vector fields and Poincaré's inequality. (2009) arXiv:0809.2872.
- [5] Bongfili A., Lanconelli E., Uguzzoni F. Stratified Lie groups and potential theory for their sub-laplacians. Springer-Verlag, Berlin-Heidelberg, 2007.
- [6] Buliga M. Dilatation structures I. Fundamentals. J. Gen. Lie Theory Appl. **1** (2) (2007) 65–95.
- [7] Buliga M. Contractible groups and linear dilatation structures. (2007) arxiv.org: 0705.1440v3.
- [8] Buliga M. Dilatation structures in sub-Riemannian geometry. (2007) arxiv.org: 0708.4298.
- [9] Buliga M. A characterization of sub-Riemannian spaces as length dilatation structures constructed via coherent projections. (2009) arxiv.org: 0810.5040v3.
- [10] Buliga M. Braided space with dilations and sub-Riemannian symmetric spaces. (2010) arxiv.org: 1005.5031v1.
- [11] Burago D. Yu., Burago Yu. D., Ivanov S. V. A Course in Metric Geometry. Graduate Studies in Mathematics, **33**, American Mathematical Society, Providence, RI, 2001.
- [12] Capogna L., Danielli D., Pauls S. D. and Tyson J. T. An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. Progress in Mathematics **259**. Birkhäuser, 2007.
- [13] Cheeger J. Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal. **9** (3) (1998) 428–517.
- [14] Christ M. Lecture on singular operators. CBMS Reg. Conf. Ser. Math., Vol. 77, Amer. Math. Soc., Providence, RI, 1990.
- [15] Le Donne E. Geodesic manifolds with a transitive subset of smooth bilipschitz maps. arxiv.org: 0804.0403v1
- [16] Van der Dries L., Goldbring I. Locally compact contractive local groups. J. of Lie Theory. **19** (2010) 685–695.
- [17] Folland G. B. Applications of analysis on nilpotent groups to partial differential equations. Bull. of Amer. Math. Soc. Vol. 83, No. 5 (1977) 912–930.
- [18] Folland G. B., Stein E. M. Hardy spaces on homogeneous groups. Princeton Univ. Press, 1982.
- [19] Gleason A. M. Groups without small subgroups. Ann. of Math. **56** (1952) 193–212.
- [20] Goldbring I. Hilbert's fifth problem for local groups. J. of Logic and Analysis. **1:5** (2009), 1–25.

- [21] Greshnov A. V. Local approximation of equiregular Carnot-Carathéodory spaces by its tangent cones. *Sib. Math. Zh.* **48** (2) (2007) 290–312.
- [22] Greshnov A. V. Applications of the group analysis of differential equations to some systems of noncommuting C^1 -smooth vector fields. *Sibirsk. Mat. Zh.* **50:1** (2009), 47–62.
- [23] Gromov M. Groups of polynomial growth and expanding maps. *Inst. Hautes Etudes Sci. Publ. Math.* **53** (1981) 53–73.
- [24] Gromov M. *Metric Structures for Riemannian and Non-Riemannian Spaces.* Birkhäuser, 2001.
- [25] Gromov M. Carno–Carathéodory spaces seen from within. *Sub-riemannian Geometry, Progress in Mathematics*, 144. Birkhäuser. (1996) 79–323.
- [26] Heinonen J. *Lectures on analysis on metric spaces.* Universitext, Springer-Verlag, New York, 2001.
- [27] Hörmander L. Hypoelliptic second order differential equations. *Acta Math.* **119** (3-4) (1967) 147–171.
- [28] Karmanova M. A New Approach to Investigation of Carnot-Caratheodory Geometry. *Doklady Mathematics* **433** (4) (2010), to appear.
- [29] Karmanova M., Vodopyanov S. Geometry of Carno-Carathéodory spaces, differentiability, coarea and area formulas. *Analysis and Mathematical Physics. Trends in Mathematics*, Birkhäuser. (2009) 233–335.
- [30] Macías R. A., Segovia C. Lipshitz functions on spaces of homogeneous type. *Adv. in Math.* **33** (1979) 257–270.
- [31] Mal'tsev A. I. On local and global topological groups. *Dokl. Akad. Nauk SSSR.* **32** (9) (1941) 606–608.
- [32] Margulis G. A., Mostov G. D. The differential of quasi-conformal mapping of a Carnot-Caratheodory spaces. *Geom. Funct. Anal.* **5** (2) (1995) 402–433.
- [33] Margulis G. A., Mostov G. D. Some remarks on definition of tangent cones in a Carnot-Caratheodory space. *J. Anal. Math.* **80** (2000) 299–317.
- [34] Mitchell J. On Carnot-Caratheodory metrics. *J. Differential Geometry* **21** (1985) 35–45.
- [35] Montanari A., Morbidelli D. Balls defined by nonsmooth vector fields and the Poincare' inequality. *Annales de l'institut Fourier.* **54**(2) (2004) 431–452.
- [36] R. Montgomery. *A Tour of Subriemannian Geometries, their Geodesics and Applications.* Providence, AMS. 2002.
- [37] Montgomery D., Zippin L. *Topological transformation groups.* Interscience, New York. 1955.

- [38] Müller-Römer P. Kontrahierbare Erweiterungen kontrahierbaren Gruppen. J. Reine Angew. Math. **283/284** (1976) 238–264.
- [39] Nagel A., Stein E.M., Wainger S. Balls and metrics defined by vector fields I: Basic properties. Acta Math. **155** (1985) 103–147.
- [40] Olver P. Non-associative local Lie groups. Journal of Lie theory **6** (1996) 23–51.
- [41] Paluszyński M., Stempak K. On quasi-metric and metric spaces // AMS Proceedings. **137** (12) (2002) 4307–4312.
- [42] Pansu P. Metriques de Carnot-Carathéodory et quasiisometries des espaces symetriques de rang un. Ann. of Math. **119** (1989) 1–60.
- [43] Petersen V. P. Gromov–Hausdorff convergence in metric space. Differential geometry: Riemannian geometry (Proc. Sympos. Pure Math., **54** Pt.3). Providence, RI: Amer. Math. Soc. (1993) 489–504.
- [44] Pontryagin L. S. Continuous Groups. Moscow, "Nauka". 1984.
- [45] Rotshild L.P., Stein E.M. Hypoelliptic differential operators and nilpotent groups. Acta Math. **137** (1976) 247–320.
- [46] Selivanova S. V. Tangent cone to a regular quasimetric Carnot–Carathéodory space. Doklady Mathematics **79** (2009) 265–269.
- [47] Selivanova S. V. Tangent cone to a quasimetric space with dilations. // Sib. Mat. J. **51** (2) (2010) 388–403.
- [48] Siebert E. Contractive automorphisms on locally compact groups. Mat. Z. **191** (1986) 73–90.
- [49] Stein E. M., Harmonic analysis: real-variables methods, orthogonality, and oscillatory integrals. Princeton, NJ, Princeton University Press. 1993.
- [50] Vodopyanov S. K. Differentiability of mappings in the geometry of Carnot manifolds. Sib. Math. Zh. **48** (2) (2007), 251–271.
- [51] Vodopyanov S. K. Geometry of Carnot–Carathéodory spaces and differentiability of mappings. Contemporary Mathematics **424** (2007), 247–302.
- [52] Vodopyanov S. K., Karmanova M. B. Local Geometry of Carnot Manifolds Under Minimal Smoothness. Doklady Mathematics **75** (2) (2007), 240–246.
- [53] Vodopyanov S. K., Karmanova M. B. Sub-Riemannian geometry under minimal smoothness of vector fields. Doklady Mathematics **78** (2) (2008), 583–588.
- [54] Vodopyanov S. K., Karmanova M. B. A Coarea Formula for Smooth Contact Mappings of Carnot Manifolds. Doklady Mathematics, **76** (4) (2007), 908–912.
- [55] Vodopyanov S. K., Karmanova M. B. An Area Formula for Contact C^1 -Mappings of Carnot Manifolds. Doklady Mathematics, **78** (2) (2008) 655–659.

- [56] Vodopyanov S. K., Karmanova M. B. An Area Formula for Contact C^1 -Mappings of Carnot Manifolds. *Complex Variables and Elliptic Equations*. **55**(1) (2010) 317–329.
- [57] Vodopyanov S. K., Selivanova S. V. Algebraic properties of the tangent cone to a quasimetric space with dilations // *Doklady Mathematics* **80** (2) (2009) 734–738.
- [58] Yandell B. H. The honor class: Hilbert’s problems and their solvers. AK Peters, Natick, Massachusetts. 2002.